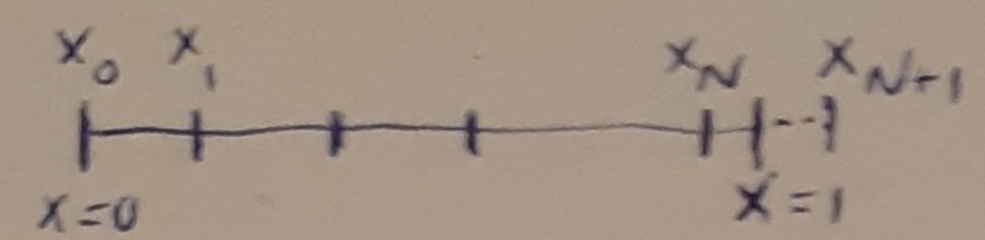
Test 2, November 2022

Exc 1 on $[0,1]$: $\frac{du}{dx} - \frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right) = \exp(x^2)$

$u(0)=1$, $\frac{du}{dx}(1) = 2$

a. Grid:



$x_j = jh$

0.2 $h = \frac{1}{N+\frac{1}{2}}$

Finite Volume Discretization

differential equation:

$$\frac{dq}{dx} = \exp(x^2) \quad \text{with } q(x) = u - (1+x^2)\frac{du}{dx}$$

0.4

$$\Rightarrow \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \frac{d}{dx}\left(u - (1+x^2)\frac{du}{dx}\right) dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \exp(x^2)\, dx$$

$$\Rightarrow u_{j+\frac{1}{2}} - (1+x_{j+\frac{1}{2}}^2)\frac{du}{dx}\Big|_{j+\frac{1}{2}} - u_{j-\frac{1}{2}} + (1+x_{j-\frac{1}{2}}^2)\frac{du}{dx}\Big|_{j-\frac{1}{2}}$$

$$= \exp(x_j^2)\, h \qquad j=1,\dots,N$$

Approximate:

0.3 $u_{j+\frac{1}{2}} = \frac{u_j + u_{j+1}}{2}$; $\frac{du}{dx}\Big|_{j+\frac{1}{2}} = \frac{u_{j+1} - u_j}{h}$

Boundary

$u_0 = 1$ (use this when $j=1$)

0.3 $\frac{u_{N+1} - u_N}{h} = 2 \Rightarrow u_{N+1} = 2h + u_N$

(use this when $j=N$)

b) $\frac{u_{j+1}+u_j}{2} - (1+x_{j+\frac{1}{2}}^2)\frac{u_{j+1}-u_j}{h} - \frac{u_j+u_{j-1}}{2} + (1+x_{j-\frac{1}{2}}^2)\frac{u_j-u_{j-1}}{h} = 0$

$\Rightarrow \frac{u_{j+1}-u_j}{2} - (1+x_{j+\frac{1}{2}}^2)\frac{u_{j+1}-u_j}{h} = -\frac{u_j-u_{j-1}}{2} - (1+x_{j-\frac{1}{2}}^2)\frac{u_j-u_{j-1}}{h}$

0.4

$\Rightarrow \left(\frac{1}{2}-\frac{1}{h}(1+x_{j+\frac{1}{2}}^2)\right)(u_{j+1}-u_j) = \left(-\frac{1}{2}-\frac{1}{h}(1+x_{j-\frac{1}{2}}^2)\right)(u_j-u_{j-1})$

$\Rightarrow u_{j+1}-u_j = \frac{-\frac{1}{2}-\frac{1}{h}(1+x_{j-\frac{1}{2}}^2)}{\frac{1}{2}-\frac{1}{h}(1+x_{j+\frac{1}{2}}^2)}(u_j-u_{j-1})$

0.2

$\Rightarrow u_{j+1}-u_j = \frac{+\frac{h}{2}+x_{j-\frac{1}{2}}^2+1}{-\frac{h}{2}+x_{j+\frac{1}{2}}^2+1}(u_j-u_{j-1})$

monotonous if $\frac{h/2+x_{j-\frac{1}{2}}^2+1}{-\frac{h}{2}+x_{j+\frac{1}{2}}^2+1} > 0$

since $h/2+x_{j-\frac{1}{2}}^2+1>0$ this is the case if $-\frac{h}{2}+1+x_{j+\frac{1}{2}}^2>0$

0.4 this is ~~[illegible]~~ the case if $1-\frac{h}{2}>0$

since $h<1$ (the grid size on interval [0,1])

this is the case

hence solution is monotonous

c) discretization

(*) $$\frac{u_{j+1}-u_{j-1}}{2h} - (1+x_{j+\frac{1}{2}}^2)\frac{u_{j+1}-u_j}{h^2} + (1+x_{j-\frac{1}{2}}^2)\frac{u_j-u_{j-1}}{h^2} = \exp(x_j^2)$$

First term

$$u_{j+1} = u_j + h u'(x_j) + \frac{h^2}{2}u''(x_j) + \frac{h^3}{6}u'''(x_j) + \cdots$$
$$u_{j-1} = u_j - h u'(x_j) + \frac{h^2}{2}u''(x_j) - \frac{h^3}{6}u'''(x_j) + \cdots$$

0.3 $$\Rightarrow \frac{u_{j+1}-u_{j-1}}{2h} = u'(x_j) + \frac{h^2}{6}u'''(x_j) + \cdots$$

$$\frac{u_{j+1}-u_{j-1}}{2h} = u'(x_j) + O(h^2)$$ second order-discretization of $\frac{du}{dx}\Big|_{x=x_j}$ 0.1

Other terms

$$-(1+x_{j+\frac{1}{2}}^2)\frac{u_{j+1}-u_j}{h^2} + (1+x_{j-\frac{1}{2}}^2)\frac{u_j-u_{j-1}}{h^2}$$
$$= -(1+x_{j+\frac{1}{2}}^2)\left(\frac{1}{h}u'(x_j) + \frac{1}{2}u''(x_j) + \frac{h}{6}u'''(x_j) + \frac{h^2}{24}u^{IV}(x_j) + \cdots\right)$$
$$+(1+x_{j-\frac{1}{2}}^2)\left(\frac{1}{h}u'(x_j) - \frac{1}{2}u''(x_j) + \frac{h}{6}u'''(x_j) - \frac{h^2}{24}u^{IV}(x_j) + \cdots\right)$$

0.3 $$= \frac{1}{h}\left[-(1+x_{j+\frac{1}{2}}^2) + (1+x_{j-\frac{1}{2}}^2)\right]u'(x_j)$$
$$+\frac{1}{2}\left[-(1+x_{j+\frac{1}{2}}^2) - (1+x_{j-\frac{1}{2}}^2)\right]u''(x_j)$$
$$+\frac{h}{6}\left[-(1+x_{j+\frac{1}{2}}^2) + (1+x_{j-\frac{1}{2}}^2)\right]u'''(x_j) + O(h^2)$$

Intermezzo:

$$(1+x_{j+\frac{1}{2}}^2) = (1+(x_j+\tfrac{1}{2}h)^2) = 1+x_j^2 + hx_j + \tfrac{1}{4}h^2$$
$$(1+x_{j-\frac{1}{2}}^2) = (1+(x_j-\tfrac{1}{2}h)^2) = 1+x_j^2 - hx_j + \tfrac{1}{4}h^2$$
$$\Rightarrow -(1+x_{j+\frac{1}{2}}^2) - (1+x_{j-\frac{1}{2}}^2) = -2(1+x_j)^2 - \tfrac{1}{2}h^2$$
0.2 $$-(1+x_{j+\frac{1}{2}}^2) + (1+x_{j-\frac{1}{2}}^2) = -2hx_j$$

$$= -2x_j u'(x_j) - (1+x_j^2)u''(x_j) - \tfrac{1}{4}h^2u''(x_j) - \tfrac{1}{3}h^2x_ju'''(x_j) + O(h^2)$$
$$= -\frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right)\Big|_{x=x_j} + O(h^2)$$ Second-order discretization of $-\frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right)\Big|_{x=x_j}$

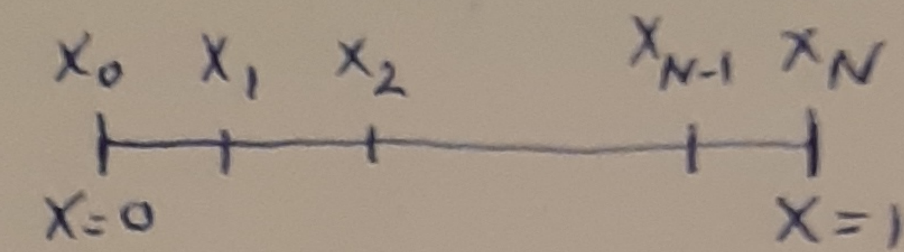
hence (*) second-order discretization of (1) 0.1

Exc 2 on $[0,1]$ $\quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - d^2 u$ $\quad t>0$, c,d constants

boundary : $u(0,t) = g(t)$ $\quad u_x(1,t) = 0$

initial : $u(x,0) = f(x)$ $\quad u_t(x,0) = 0$

a) Grid:



$x_j = j\Delta x$

$\Delta x = \frac{1}{N}$

0.3

Finite - difference discretization , $u_j(t) \approx u(x_j, t)$

0.3 $\quad \frac{d^2 u_j}{dt^2} = c^2 \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} - d^2 u_j$ $\quad j = 1, \dots, N$

Boundary conditions

$u_0(t) = g(t)$ $\quad$ use when $j=1$

0.3 $\quad \frac{u_{N+1}(t) - u_{N-1}(t)}{2h} = 0 \Rightarrow u_{N+1}(t) = u_{N-1}(t)$ $\quad$ use when $j=N$

Initial conditions

$u_j(0) = f(x_j)$

0.3 $\quad \frac{du_j}{dt}(0) = 0$ $\quad j = 1, \dots, N$

b) $j=1$ $\quad \frac{d^2 u_1}{dt^2} = c^2 \frac{u_2 - 2u_1}{\Delta x^2} - d^2 u_1 + \frac{c^2}{h^2} g(t)$

$j=N$ $\quad \frac{d^2 u_N}{dt^2} = c^2 \frac{-2u_N + 2u_{N-1}}{\Delta x^2} - d^2 u_N$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \quad b = \frac{c^2}{\Delta x^2} \begin{bmatrix} g(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A = \frac{c^2}{\Delta x^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{bmatrix} - d^2 I$$

0.1 $\quad$ 0.1 $\quad$ 0.2

$$\vec{u}(0) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix} \qquad \frac{d}{dt}\vec{u}(0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

0.2 $\quad$ 0.2

c) Typical row of A

0.2 $L_{\Delta x} u_j = \frac{c^2}{\Delta x^2} (u_{j+1} - 2u_j + u_{j-1}) - d^2 u_j$

set $u_j = e^{ij\theta}$ (with $i = \sqrt{-1}$)

$$L_{\Delta x} e^{ij\theta} = \frac{c^2}{\Delta x^2} (e^{i(j+1)\theta} - 2e^{ij\theta} + e^{i(j-1)\theta}) - d^2 e^{ij\theta}$$

0.4 $$= \left[\frac{c^2}{\Delta x^2} (e^{i\theta} - 2 + e^{-i\theta}) - d^2 \right] e^{ij\theta}$$

estimate of eigenvalue of A:

0.4 $$\lambda \approx \frac{c^2}{\Delta x^2} (e^{i\theta} + e^{-i\theta} - 2) - d^2 = -\frac{4c^2}{\Delta x^2} \sin^2 \frac{\theta}{2} - d^2$$

(note $e^{i\theta} + e^{-i\theta} - 2 = 2\cos\theta - 2 = 2(\cos\theta - 1) = -4\sin^2\frac{\theta}{2}$)

d) Define $v = \frac{du}{dt}$

0.1 $$\Rightarrow \frac{dv}{dt} = Au + b(t)$$

0.2 $$\Rightarrow \frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ b(t) \end{bmatrix} \text{ with } \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = w(0) = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

0.1 $$w = \begin{bmatrix} u \\ v \end{bmatrix}, \quad B = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}, \quad h(t) = \begin{bmatrix} 0 \\ b(t) \end{bmatrix}$$

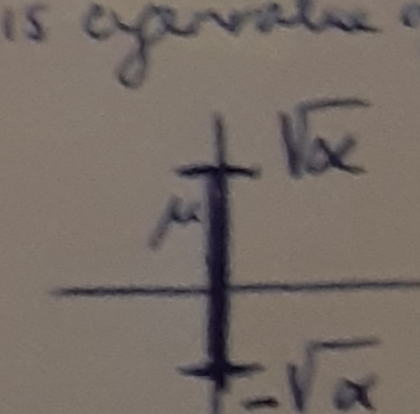
e) • $\det\left(\begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix} - \mu I\right) = 0 \Rightarrow \det\begin{pmatrix} -\mu & 1 \\ \lambda & -\mu \end{pmatrix} = 0$

0.4 $\Rightarrow \mu^2 - \lambda = 0 \Rightarrow \mu = \pm\sqrt{\lambda}$ (we know λ real) $\lambda < 0$

note: if $Au = \lambda u$, $u \neq 0$ $\Rightarrow \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ \lambda I & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow$ eigenvalue μ of B, is eigenvalue of $\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$

• Assume $-\alpha \le \lambda \le 0$ for some $\alpha > 0$

$\Rightarrow -i\sqrt{\alpha} \le \mu \le i\sqrt{\alpha} \Rightarrow$



0.3 eigenvalues μ on imaginary axis

• $\lambda \approx -\frac{4c^2}{\Delta x^2} \sin^2\frac{\theta}{2} - d^2 \Rightarrow \lambda \in \left[-\frac{4c^2}{\Delta x^2} - d^2, 0\right] \Rightarrow \alpha \approx \frac{4c^2}{\Delta x^2} + d^2$

BT

0.3

f) $y' = f(t,y)$, time-integration $w_{n+1} = w_n + \Delta t\, f(t_{n+1}, w_n + \Delta t\, f(t, w_n))$

i) apply method to test equation

$$y' = \lambda y \qquad (\text{i.e } f(t,y) = \lambda y)$$

0.1

0.1 $\Rightarrow w_{n+1} = w_n + \Delta t \lambda (w_n + \Delta t \lambda w_n) = (1 + \Delta t \lambda + (\Delta t \lambda)^2) w_n$

setting $z = \Delta t \lambda$

0.1

$\Rightarrow w_{n+1} = (1 + z + z^2) w_n \quad \Rightarrow$ amplification factor: $1 + z + z^2$

ii). take z on imaginary axis : $z = iy$, y real

$$g(z) = g(iy) = 1 + (iy) + (iy)^2 = 1 - y^2 + iy$$

0.1 $\Rightarrow |g(z)|^2 = (1-y^2)^2 + y^2 = 1 - 2y^2 + y^4 + y^2 = 1 - y^2 + y^4$

• $|g(z)| \leq 1$ if $1 - y^2 + y^4 \leq 1$, i.e if $y^2(y^2-1) \leq 0$,

0.1 $\quad y^2 - 1 \leq 0, \quad -1 \leq y \leq 1$

iii) eigenvalue μ of B is $\mu = \pm\sqrt{\lambda} = \pm i\sqrt{|\lambda|}$

with $\lambda \in \left[-\frac{4c^2}{\Delta x^2} - d^2, 0\right] = \left[-\frac{4}{\Delta x^2} - 16, 0\right]$ if $c=1$, $d=4$

0.1

using ii) : time integration stable if

0.2

$$\underbrace{\sqrt{\frac{4}{\Delta x^2} + 16}}_{|\lambda_{max}|}\, \Delta t \leq 1 \quad \Rightarrow \quad \Delta t \leq \frac{1}{2\sqrt{\frac{1}{\Delta x^2} + 4}}$$

g) i) $u(t) = v \exp(i 2\pi f t)$ v eigenvector of A , with eigenvalue λ

$$\frac{d^2 u}{dt^2} = Au \Rightarrow (i2\pi f)^2 v \exp(i2\pi f t) = Av \exp(i2\pi f t)$$

0.2

$$\Rightarrow (i2\pi f)^2 v = \lambda v \quad \Rightarrow \quad -(2\pi f)^2 v = \lambda v$$

$$\Rightarrow f = \pm \frac{\sqrt{-\lambda}}{2\pi}$$

ii) $b(t) = v \exp(i\omega t)$ v eigenvector of A

take $u(t) = \alpha v \exp(i\omega t)$: $\alpha v (i\omega)^2 \exp(i\omega t) \stackrel{?}{=} A\alpha v \exp(i\omega t) + b(t)$

0.2

$$\Rightarrow -\alpha\omega^2 v \exp(i\omega t) \stackrel{?}{=} \alpha\lambda v \exp(i\omega t) + v \exp(i\omega t)$$

$$\Rightarrow -\alpha\omega^2 v \stackrel{?}{=} (\alpha\lambda + 1) v$$

$u(t)$ solution if $\alpha = \frac{-1}{\omega^2 + \lambda}$ 0.5

exercise should have stated this

0.2 iii) $\omega^2 + \lambda = 0$ for $\omega = \pm\sqrt{-\lambda}$, i.e $\alpha \to \pm\infty$ if $\omega \to \pm\sqrt{-\lambda}$

Note : if $\omega \to \pm\sqrt{\lambda}$ then $\alpha \to \frac{-1}{2\lambda}$